



MIXED HESSIANS CLASS OF m -CONVEX FUNCTIONS

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ABSTRACT. In this article, we study mixed Hessians introduced by Trudenger-Wong to introduce capacitive quantities in the class of m -convex functions.

Keywords. strongly m -subharmonic function, m -convex function, Mixed Hessian, Borel measures, Hessians.

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1. INTRODUCTION

As is known, strongly m -subharmonic (sh_m) functions have a practically useful connection with m -convex (m -cv) functions. It turned out to be a very convenient technique for studying m -cv functions. The purpose of this work is to introduce mixed Hessians in the class of m -cv functions, using this connection.

Recall, that the function $u(x) \in C^2(D)$ defined in the area $D \subset \mathbb{R}^n$, is called m -convex, $1 \leq m \leq n$, if

$$H^k(u) = H^k(\lambda) = \sum_{1 \leq j_1 < \dots < j_k \leq n} \lambda_{j_1} \dots \lambda_{j_k}$$

is a Hessians of the eigenvalues of $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ an orthogonal matrix

$\left(\frac{\partial^2 u}{\partial x_j \partial x_k} \right)$ at each point $x \in D$ it satisfies the following condition

$$m\text{-cv} \cap C^2(D) = \{H^k(u) \geq 0, k = 1, \dots, n - m + 1\}. \quad (1)$$

Similarly, a twice smooth function $u(z) \in C^2(D)$ defined in the domain $D \subset \mathbb{C}^n$ is called strongly m -subharmonic $u(z) \in sh_m(D)$, if the differential forms (currents)

$$(dd^c u)^k \wedge \beta^{n-k} \geq 0, k = 1, \dots, n - m + 1. \quad (2)$$



Here $\beta = dd^c |z|^2$ – the fundamental form of volume in \mathbb{C}^n .

The theory sh_m – of functions is well developed and currently the subject of study by many mathematicians (Z. Blotsky [Bl] , S. Dinev and S. Kolodziej [DK 1, DK 2], S. Li [Li] , H.C. Lu [Lu 1], [Lu 2] etc.). A sufficiently comprehensive overview of this theory can be found in the article by A. Sadullaev and B. Abdullaev [AS], published in the proceedings of the Moscow Institute of the Russian Academy of Sciences.

Operators $(dd^c u)^k \wedge \beta^{n-k}$ closely related to the Hessians. For a twice smooth function $u \in C^2(D)$ the second order differential $dd^c u = \frac{i}{2} \sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k$ (at

a fixed point $o \in D$) is a Hermitian quadratic form. After an appropriate unitary transformation of the coordinates, it is brought to a diagonal form

$dd^c u = \frac{i}{2} [\lambda_1 dz_1 \wedge d\bar{z}_1 + \dots + \lambda_n dz_n \wedge d\bar{z}_n]$, where $\lambda_1, \dots, \lambda_n$ – the eigenvalues of the

Hermitian matrix $\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right)$ are real: $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$. It is worth noting that a

unitary transformation does not change the differential form $\beta = dd^c \|z\|^2$. It is easy to see that

$$(dd^c u)^k \wedge \beta^{n-k} = k!(n-k)! H^k(u) \beta^n, \quad (3)$$

where is $H^k(u) = \sum_{1 \leq j_1 < \dots < j_k \leq n} \lambda_{j_1} \dots \lambda_{j_k}$ – the Hessian of the dimension k of the vector

$$\lambda = \lambda(u) \in \mathbb{R}^n.$$

Consequently, a twice smooth function $u(z) \in C^2(D)$, $D \subset \mathbb{C}^n$, is strongly m – subharmonic if $o \in D$ the inequalities hold at each point

$$H^k(u) = H_o^k(u) \geq 0, \quad k = 1, 2, \dots, n - m + 1.$$

Note that the concept of m – a subharmonic function in a generalized sense is also defined in the general case for upper semi-continuous functions.

Definition 1. A function $u(z)$, defined in a domain $D \subset \mathbb{C}^n$ is called sh_m , if it is upper semi-continuous and for any twice smooth sh_m functions $v_1, \dots, v_{n-m} \in C^2(D) \cap sh_m(D)$ the current $dd^c u \wedge dd^c v_1 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1}$ is defined as

$$\begin{aligned} & \left[dd^c u \wedge dd^c v_1 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \right](\omega) = \\ & = \int u dd^c v_1 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega, \quad \omega \in F^{0,0} \end{aligned} \quad (4)$$

✚ positive



$$\int u dd^c v_1 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega \geq 0, \quad \forall \omega \in F^{0,0}, \omega \geq 0.$$

Here $F^{0,0}(D)$ is a family of infinitely smooth, finitely supported D functions.

In the work of Blotsky [B1], it was proven that this definition is correct, that for functions $u \in C^2(D)$ this definition coincides with the original definition sh_m – of functions. Moreover, in the class of bounded sh_m – functions, operators are defined $(dd^c u)^k \wedge \beta^{n-k} \geq 0, k = 1, 2, \dots, n - m + 1$ as Borel measures in the domain D (see [B1], [AS]). According to (3), in the class of bounded sh_m – functions the Hessians $H^k(u), k = 1, 2, \dots, n - m + 1$ are Borel measures in D .

The key point studying $m - cv$ functions in a broader class of functions is the following connection $m - cv$ and sh_m functions (see [SI 1]). We put \mathbb{R}_x^n in $\mathbb{C}_z^n, \mathbb{R}_x^n \subset \mathbb{C}_z^n = \mathbb{R}_x^n + i\mathbb{R}_y^n (z = x + iy)$, as a real $n -$ dimensional subspace of the complex space \mathbb{C}_z^n .

Theorem 1. A twice smooth function $u(x) \in C^2(D), D \subset \mathbb{R}_x^n$, is $m - cv$ in D if and only if a function $u^c(z) = u^c(x + iy) = u(x)$, that does not depend on variables $y \in \mathbb{R}_y^n$ is sh_m in the domain $D \times i\mathbb{R}_y^n$.

Theorem 1 allows us to define a $m - convex$ function in the class of upper semi-continuous function

Definition 2. An upper semi-continuous function $u(x)$ in a domain $D \subset \mathbb{R}_x^n$ is called $m - convex$ D , if the function $u^c(z)$ is strongly $m - subharmonic$, $u^c(z) \in sh_m(D \times i\mathbb{R}_y^n)$.

2. DEFINITIONS OF A MIXED HESSIAN.

Theorem 1 allows us to define the Hessians $H^k(u), k = 1, 2, \dots, n - m + 1$ in the class of bounded $m - cv$ functions $u(x) \in m - cv(D)$ as Borel measures in D . By using the properties of $sh_m(D \times i\mathbb{R}_y^n)$ functions in [SASh] a number of properties of Hessians were proven $H^k(u), k = 1, 2, \dots, n - m + 1$.

The purpose of this work is to investigate the so-called mixed Hessian, introduced by Trudinger-Wong [TW 3] to introduce capacitive quantities in the class $u(x) \in m - cv(D)$. As it turned out, mixed Hessians are related to differential operators

$$dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k \wedge \beta^{m-1} \geq 0, \quad k = 1, 2, \dots, n - m + 1,$$



for various functions $u_1(z), u_2(z), \dots, u_k(z) \in sh_m(D)$. In the class $sh_m(D) \cap L_{loc}^\infty(D)$, $1 \leq m \leq n$, we define differential operators $dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k \wedge \beta^{m-1} \geq 0$ as currents (generalized functions) of bidegree (k, k) , recurrence relation according to k .

$k = 1$. The relationship

$$dd^c u_1 \wedge \beta^{m-1} \wedge \alpha = \int_D u_1(z) \beta^{m-1} \wedge dd^c \alpha, \quad \alpha \in F^{n-m, n-m}(D),$$

defines the current of bi-degree (m, m) . It is positive because if $u_1^j(z) \downarrow u_1(z)$ approximation $u_1(z)$ smooth $u_1^j(z) \in sh_m \cap C^\infty$ functions, then $dd^c u_1^j \wedge \beta^{m-1} \wedge \alpha \geq 0 \quad \forall \alpha \geq 0$. From here,

$$dd^c u_1^j \wedge \beta^{m-1} \wedge \alpha = \int_D u_1^j(z) \beta^{m-1} \wedge dd^c \alpha \geq 0, \quad \forall \alpha \in F^{n-m, n-m}(D), \alpha \geq 0,$$

when $j \rightarrow \infty$ it gives us

$$\int_D u_1(z) \beta^{m-1} \wedge dd^c \alpha \geq 0, \quad \forall \alpha \in F^{n-m, n-m}(D), \alpha \geq 0.$$

From the positivity of current $dd^c u_1 \wedge \beta^{m-1}$ it follows that it is a current of measure type.

Let us now assume recursively that the operator

$$dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_{k-1} \wedge \beta^{m-1}$$

is a positive current of measure type, we will show that $dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k \wedge \beta^{m-1}$ is also a positive current of measure type. Since $dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_{k-1} \wedge \beta^{m-1}$ is a positive current of measure type, then the relationship

$$\begin{aligned} & dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k \wedge \beta^{m-1} \wedge \alpha = \\ & = \int_D u_k(z) dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_{k-1} \wedge \beta^{m-1} \wedge dd^c \alpha, \quad \alpha \in F^{n-m-k+1, n-m-k+1}(D), \end{aligned}$$

defined bi-degree current $(n-m-k+1, n-m-k+1)$, because by the recurrence relation $dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_{k-1} \wedge \beta^{m-1} \wedge dd^c \alpha$ it is a positive current of measure type. Let us show that the resulting current is positive. If $u_i^j(z) \downarrow u_i(z)$, $i = 1, 2, \dots, k$, approximation $u_i(z)$ is smooth with $u_i^j(z) \in sh_m \cap C^\infty$



functions, then currents $dd^c u_1^j \wedge dd^c u_2^j \wedge \dots \wedge dd^c u_k^t \wedge \beta^{m-1}$, $j \in \mathbb{N}$, $t \in \mathbb{N}$ are positive, because $k \leq n - m + 1$, $dd^c u_1^j \wedge dd^c u_2^j \wedge \dots \wedge dd^c u_k^t \wedge \beta^{m-1} \wedge \alpha \geq 0 \quad \forall \alpha \geq 0$. From here,

$$dd^c u_1^j \wedge dd^c u_2^j \wedge \dots \wedge dd^c u_k^t \wedge \beta^{m-1} \wedge \alpha = \int_D u_k^t(z) dd^c u_1^j \wedge dd^c u_2^j \wedge \dots \wedge dd^c u_{k-1}^j \wedge \beta^{m-1} \wedge dd^c \alpha \geq 0,$$

$$\forall \alpha \in F^{n-m-k+1, n-m-k+1}(D), \alpha \geq 0.$$

when $j \rightarrow \infty$ it gives us

$$\int_D u_k^t(z) dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_{k-1} \wedge \beta^{m-1} \wedge dd^c \alpha \geq 0,$$

$$\forall \alpha \in F^{n-m-k+1, n-m-k+1}(D), \alpha \geq 0.$$

In turn, at $t \rightarrow \infty$ hence we obtain the positivity of the current

$$dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k \wedge \beta^{m-1} \wedge \alpha = \int_D u_k(z) dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_{k-1} \wedge \beta^{m-1} \wedge dd^c \alpha \geq 0,$$

$$\forall \alpha \in F^{n-m-k+1, n-m-k+1}(D), \alpha \geq 0.$$

From the positivity of this current follows the important fact that

$$dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k \wedge \beta^{m-1} \geq 0, \quad k = 1, 2, \dots, n - m + 1, \quad (5)$$

for various functions $u_1(z), u_2(z), \dots, u_k(z) \in sh_m(D)$ are defined in the class $sh_m(D) \cap L_{loc}^\infty(D)$, $1 \leq m \leq n$, currents (generalized functions) of bi-degree (k, k) , type of measure.

Note that for twice smooth vector-functions $\vec{u} = (u_1, u_2, \dots, u_k) \in sh_m(D) \cap C^2(D)$ the differential form $dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k \wedge \beta^{n-k}$, $k = 1, 2, \dots, n - m + 1$, is positive definite, $dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k \wedge \beta^{n-k} \geq 0$.

Let us denote $dd^c u_t = \frac{i}{2} \sum_{j,s} \frac{\partial^2 u_t}{\partial z_j \partial \bar{z}_s} dz_j \wedge d\bar{z}_s$, $t = 1, 2, \dots, k$. By $\lambda_j^t = \frac{\partial^2 u_t}{\partial z_j \partial \bar{z}_j}$ the

diagonal elements of the matrix $\left(\frac{\partial^2 u_t}{\partial z_j \partial \bar{z}_s} \right)$. Then

$$dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k \wedge \beta^{n-k} = C_{k,n} \left[\sum_{1 \leq j_1 \neq j_2 \neq \dots \neq j_k \leq n} \lambda_{j_1}^1 \lambda_{j_2}^2 \dots \lambda_{j_k}^k \right] \beta^n$$



where $C_{k,n} > 0$ is the constant. Value

$$H^k(\vec{u}) = \sum_{1 \leq j_1 \neq j_2 \neq \dots \neq j_k \leq n} \lambda_{j_1}^1 \lambda_{j_2}^2 \dots \lambda_{j_k}^k$$

is called mixed Hessian for functions $u_1, u_2, \dots, u_k \in sh_m(D) \cap C^2(D)$. Thus,

$$dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k \wedge \beta^{n-k} = C_{k,n} H^k(\vec{u}) \beta^n \quad (6)$$

and from positivity $dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k \wedge \beta^{n-k} \geq 0$ it follows that mixed

Hessian is positive: $H^k(\vec{u}) \geq 0$.

3. GENERAL DEFINITIONS OF A MIXED HESSIAN.

By using Theorem 1 in this section we define mixed Hessians $H^k(u_1, u_2, \dots, u_k)$, $k = 1, 2, \dots, n - m + 1$, for various functions $u_1(x), u_2(x), \dots, u_k(x) \in m - cv(D) \cap L_{loc}^\infty(D)$, $D \subset \mathbb{R}^n$.

We put as before \mathbb{R}_x^n in \mathbb{C}_z^n , $\mathbb{R}_x^n \subset \mathbb{C}_z^n = \mathbb{R}_x^n + i\mathbb{R}_y^n$ ($z = x + iy$), as a real n -dimensional subspace of the complex space \mathbb{C}_z^n , we find sh_m in the domain $D \times i\mathbb{R}_y^n$ of the function

$$u_1^c(z), u_2^c(z), \dots, u_k^c(z) \in sh_m(D \times i\mathbb{R}^n) \cap L_{loc}^\infty(D \times i\mathbb{R}^n), D \times i\mathbb{R}^n \subset \mathbb{C}^n.$$

Then, according to (5), $dd^c u_1^c \wedge dd^c u_2^c \wedge \dots \wedge dd^c u_k^c \wedge \beta^{n-k} \geq 0$, $k = 1, 2, \dots, n - m + 1$, represents a current (generalized function) in the domain $D \times i\mathbb{R}_y^n \subset \mathbb{C}_z^n$. If $u_{ij}^c(z) = u_{ij}^c \circ K(w - z)$, $i = 1, 2, \dots, k$, $j = 1, 2, \dots$ the standard approximation, then smooth sequences $u_{ij}^c(z) \downarrow u_i^c(z)$, for $j \rightarrow \infty$, $i = 1, 2, \dots, k$. Moreover, there is a weakly convergence of currents,

$$\begin{aligned} dd^c u_{1j}^c \wedge dd^c u_{2j}^c \wedge \dots \wedge dd^c u_{kj}^c \wedge \beta^{n-k} &\mapsto \\ \mapsto dd^c u_1^c \wedge dd^c u_2^c \wedge \dots \wedge dd^c u_k^c \wedge \beta^{n-k}, &k = 1, 2, \dots, n - m + 1. \end{aligned} \quad (7)$$

(6) and (7) entails the convergence of the Hessian

$$H^k(u_{1j}^c, u_{2j}^c, \dots, u_{kj}^c) \mapsto H^k(u_1^c, u_2^c, \dots, u_k^c), k = 1, 2, \dots, n - m + 1. \quad (8)$$

(8) defines for $u_1^c, u_2^c, \dots, u_k^c \in sh_m(D \times i\mathbb{R}_y^n) \cap L_{loc}^\infty(D \times i\mathbb{R}_y^n)$ Hessians $H^k(u_1^c, u_2^c, \dots, u_k^c)$, $k = 1, 2, \dots, n - m + 1$, as Borel measures, $H^k(u_1^c, u_2^c, \dots, u_k^c) = \mu^k$.



Since $u_1^c, u_2^c, \dots, u_k^c \in sh_m(D \times i\mathbb{R}_y^n)$ they do not depend on $y \in \mathbb{R}_y^n$, then for any Borel sets $E_x \subset D, E_y \subset \mathbb{R}_y^n$ the measures $\frac{4^k}{mes E_y} \mu_k(E_x \times E_y)$ do not depend on the set $E_y \subset \mathbb{R}_y^n$, i.e. $\frac{4^k}{mes E_y} \mu_k(E_x \times E_y) = \nu_k(E_x)$. Borel measures $\nu_k : \nu_k(E_x) = \frac{4^k}{mes E_y} \mu_k(E_x \times E_y), k = 1, 2, \dots, n - m + 1,$ (9)

it's natural to call mixed Hessian $H^k(u_1, u_2, \dots, u_k), k = 1, 2, \dots, n - m + 1,$ for bounded, $m -$ convex functions $u_1, u_2, \dots, u_k \in m - cv(D).$

4. ESTIMATION OF THE MIXED HESSIAN ON AVERAGE.

To have a metric characteristic of sets $E \subset D$ associated with capacity, we need to introduce the corresponding capacity quantity

If $\Omega = \{\rho(z) < 0\} \subset \mathbb{C}^n -$ the domain is strictly pseudoconvex, $\sigma = \min_G \rho(\alpha),$ then for any $\sigma < r < 0$ and for any function $u \in sh_m(\Omega) \cap C^2(\bar{\Omega})$ the next integral formula holds (see [S3])

$$\int_{\sigma}^r dt \int_{\rho(z) \leq t} (dd^c \rho)^{n-k} \wedge (dd^c u)^k \leq (M - m) \int_{\rho(z) \leq r} (dd^c \rho)^{n-k+1} \wedge (dd^c u)^{k-1},$$

where $1 \leq k \leq n - m + 1, M = \sup_{\Omega} u(z), m = \inf_{\Omega} u(z).$

In particular, for the function $u(z) \in sh_m(B) \cap C^2(\bar{B}), 0 \leq u \leq 1,$ in a unit ball $B_z = \{z \in \mathbb{C}^n : \rho(z) < 1\}, \rho(z) = |z|^2 - 1,$ we have

$$\int_{-1}^r dt \int_{|z|^2 \leq 1+t} \beta^{n-k} \wedge (dd^c u)^k \leq \int_{|z|^2 \leq r} \beta^{n-k+1} \wedge (dd^c u)^{k-1}. \tag{10}$$

Formula (10) has a generalization. Let $\bar{u} = (u_1, u_2, \dots, u_k) \in sh_m(B) \cap C^2(\bar{B}), 0 \leq u_j \leq 1, j = 1, 2, \dots, k.$ Then, by the Stokes formula



$$\begin{aligned}
 & \int_{-1}^r dt \int_{|z|^2 \leq 1+t} \beta^{n-k} \wedge dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k = \\
 & = \int_{-1}^r dt \int_{|z|^2 = 1+t} \beta^{n-k} \wedge d^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k = \\
 & = \int_{|z|^2 \leq 1+r} \beta^{n-k} \wedge d\rho \wedge d^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k = \\
 & = \int_{|z|^2 \leq 1+r} \beta^{n-k} \wedge du_1 \wedge d^c \rho \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k = \\
 & = \int_{|z|^2 \leq 1+r} d \left[\beta^{n-k} \wedge u_1 d^c \rho \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k \right] - \\
 & - \int_{|z|^2 \leq 1+r} u_1 \beta^{n-k} \wedge dd^c \rho \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k = \\
 & = \int_{|z|^2 = 1+r} u_1 d^c |z^2| \wedge \left(dd^c |z^2| \right)^{n-k} \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k - \\
 & - \int_{|z|^2 \leq 1+r} u_1 \beta^{n-k+1} \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k.
 \end{aligned}$$

Since here $0 \leq u_1 \leq 1$ the integrand differential forms are also positive, then

$$\begin{aligned}
 & \int_{-1}^r dt \int_{|z|^2 \leq 1+t} \beta^{n-k} \wedge dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k \leq \\
 & \leq \int_{|z|^2 = 1+r} d^c |z^2| \wedge \left(dd^c |z^2| \right)^{n-k} \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k = \\
 & = \int_{|z|^2 \leq 1+r} \beta^{n-k+1} \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k.
 \end{aligned}$$

Repeating this procedure k – once, we obtain

$$\int_{-1}^{r_1} dr_1 \int_{-1}^{r_2} dr_2 \dots \int_{-1}^{r_k} dr_k \int_{|z|^2 \leq 1+r_k} dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k \wedge \beta^{n-k} \leq \int_{|z|^2 \leq 1+r_1} \beta^n = Vol(B_z).$$

If we move here to mixed Hessians (formula (6)), we obtain



$$\begin{aligned} & \int_{-1}^0 dr_1 \int_{-1}^{r_1} dr_2 \dots \int_{-1}^{r_{k-1}} dr_k \int_{|z|^2 \leq 1+r_k} dd^c u_1 \wedge dd^c u_2 \wedge \dots \wedge dd^c u_k \wedge \beta^{n-k} = \\ & = \int_{-1}^0 dr_1 \int_{-1}^{r_1} dr_2 \dots \int_{-1}^{r_{k-1}} dr_k \int_{|z|^2 \leq 1+r_k} H^k(\vec{u}) \beta^n \leq \frac{Vol(B_z)}{C_{k,n}}. \end{aligned} \tag{11}$$

The left side of (11) can be estimated from below. For $-1 < r_0 < 0$

$$\begin{aligned} & \int_{-1}^0 dr_1 \int_{-1}^{r_1} dr_2 \dots \int_{-1}^{r_{k-1}} dr_k \int_{|z|^2 \leq 1+r_k} H^k(\vec{u}) \beta^n \geq \int_{r_0}^1 dr_1 \int_{r_0}^{r_1} dr_2 \dots \int_{r_0}^{r_{k-1}} dr_k \int_{|z|^2 \leq 1+r_0} H^k(\vec{u}) \beta^n = \\ & = \int_{|z|^2 \leq 1+r_0} H^k(\vec{u}) \beta^n \int_{r_0}^0 dr_1 \int_{r_0}^{r_2} dr_2 \dots \int_{r_0}^{r_{k-1}} dr_k = \frac{(-r_0)^k}{k!} \int_{|z|^2 \leq 1+r_0} H^k(\vec{u}) \beta^n. \end{aligned}$$

Hence, assuming $1 + r_0 = r$, $0 < r < 1$,

$$\int_{|z|^2 \leq r} H^k(\vec{u}) \beta^n \leq \frac{Vol(B_z) k!}{C_{k,n} (1-r)^k}, \quad 1 \leq k \leq n - m + 1, \quad 0 < r < 1. \tag{12}$$

Note that formula (12) is also true for the vector-function

$$\vec{u}(z) = (u_1(z), u_2(z), \dots, u_k(z)) \in sh_m(B) \cap L^\infty(B).$$

We now turn to the m -convex functions

$$\vec{u}(x) = (u_1(x), u_2(x), \dots, u_k(x)) \in m-cv(B) \cap L^\infty(B).$$

As above we put \mathbb{R}_x^n in \mathbb{C}_z^n , $\mathbb{R}_x^n \subset \mathbb{C}_z^n = \mathbb{R}_x^n + i\mathbb{R}_y^n$ ($z = x + iy$) and build elevations $\vec{u}(x) = (u_1(x), u_2(x), \dots, u_k(x))$ into the domain $\Omega = B \times i\mathbb{R}^n$, we find

$$\vec{u}^c(z) = (u_1^c(z), u_2^c(z), \dots, u_k^c(z)) \in sh_m(\Omega) \cap L^\infty(\Omega),$$

where $\vec{u}^c(z) = (u_1(x), u_2(x), \dots, u_k(x))$, $x \in B, y \in \mathbb{R}^n$. We apply formula (12)

to the vector function $\vec{u}^c(z)$.

$$\int_{|z|^2 \leq r} H^k(\vec{u}^c) dV(z) \leq \frac{Vol(B_z) k!}{C_{k,n} (1-r)^k}, \quad 1 \leq k \leq n - m + 1, \quad 0 < r < 1.$$

Here, we transition from a multiple integral to an iterated integral

$$\int_{|y|^2 \leq r} dV(y) \int_{|x|^2 \leq 1-|y|^2} H^k(\vec{u}^c) dV(x) \leq \frac{Vol(B_z) k!}{C_{k,n} (1-r)^k} \tag{13}$$

using Fubini's formula.



Note that for twice smooth functions $u(x) \in C^2(D)$, $D \subset \mathbb{R}^n$, the relationship $\left(\frac{\partial^2 u^c(z)}{\partial z_j \partial \bar{z}_k}\right) = \frac{1}{4} \left(\frac{\partial^2 u(x)}{\partial x_j \partial x_k}\right)$ is correct. Hence, the eigenvalues of the matrix $\left(\frac{\partial^2 u^c(z)}{\partial z_j \partial \bar{z}_k}\right), \left(\frac{\partial^2 u(x)}{\partial x_j \partial x_k}\right)$ are related by the relations $(\lambda_1^c(z), \dots, \lambda_n^c(z)) = \frac{1}{4}(\lambda_1(x), \dots, \lambda_n(x))$. Therefore, for the mixed Hessian $H^k(\bar{u}) = \sum_{1 \leq j_1 \neq j_2 \neq \dots \neq j_k \leq n} \lambda_{j_1}^1 \lambda_{j_2}^2 \dots \lambda_{j_k}^k$ we have $H^k(\bar{u}^c) = \frac{1}{4^k} H^k(\bar{u})$. Since the vector-function $\bar{u}^c(z)$ does not depend on $y \in \mathbb{R}^n$, then according to (13)

$$(14) \quad \int_{|y|^2 \leq r} dV(y) \int_{|x|^2 \leq 1-|y|^2} H^k(\bar{u}) dV(x) \leq \frac{4^k \text{Vol}(B_z) k!}{C_{k,n} (1-r)^k}, 1 \leq k \leq n-m+1, 0 < r < 1.$$

The left side of formula (14) expresses a kind of local averaging of Hessians $H^k(\bar{u})$, $1 \leq k \leq n-m+1$, for m -convex functions. In particular, the following holds

Theorem 2. In the class of locally uniformly bounded functions, $L = \{u(x) \in m\text{-cv}(D) \cap L_{loc}^\infty(D)\}$, the family of integrals

$$\int_K H^k(\bar{u}, x) dV(x), \bar{u} = (u_1(x), u_2(x), \dots, u_k(x)) \in L,$$

uniformly bounded for any compact $K \subset D$, $1 \leq k \leq n-k+1$.

The class $L = \{u(x) \in m\text{-cv}(D) \cap L_{loc}^\infty(D)\}$ also has weakly convergence with mixed Hessians

Theorem 3. If the sequences $u_{tj}(x) \in L$, $t=1, 2, \dots, k$, $j=1, 2, \dots$, are decreasing $u_{tj}(x) \downarrow u_t(x) \in L$, then the sequence of Borel measures $H^k(\bar{u}_j)$ weakly converges to $H^k(\bar{u})$, $H^k(\bar{u}_j) \mapsto H^k(\bar{u})$, $k=1, 2, \dots, n-m+1$.

Proof Theorems similarly follow from the corresponding result for the class $L^c = \{u(z) \in sh_m(D) \cap L_{loc}^\infty(D)\}$.



$$\mathcal{P}_{m_{cv}}(E, D) = - \int_D \omega^*(x, E, D) dV$$

is called the $\mathcal{P}_{m_{cv}}$ -capacity of the set E with respect to D .

Thus, the $\mathcal{P}_{m_{cv}}$ -capacitance expresses the capacitive value of the pair. (E, D) .

Such a pair is usually called a capacitor in \mathbb{R}^n .

$\mathcal{P}_{m_{cv}}$ - the capacity has the following properties:

1. $\mathcal{P}_{m_{cv}}(E, D) = - \int_D \omega(x, E, D) dV$;

It follows from the fact that the set $\{x \in D : \omega(x, E, D) < \omega^*(x, E, D)\}$ has zero Lebesgue measure.

2. $\mathcal{P}_{m_{cv}}(E, D) \geq 0$ and $\mathcal{P}_{m_{cv}}(E, D) = 0$ if and only if E is a m_{cv} polar set in D .

Let us present some simple properties of the $\mathcal{P}_{m_{cv}}$ -measure.

1. (monotony) If $E_1 \subset E_2$, then $\omega^*(x, E_1, D) \geq \omega^*(x, E_2, D)$; if $E \subset D_1 \subset D_2$, then $\omega^*(x, E, D_1) \geq \omega^*(x, E, D_2)$. Follows from class embeddings $\mathcal{U}(E_1, D) \supset \mathcal{U}(E_2, D)$, $\mathcal{U}(E, D_1) \supset \mathcal{U}(E, D_2)$;

2. $\omega^*(x, U, D) \in \mathcal{U}(U, D)$ for open sets $U \subset D$ and therefore $\omega^*(x, U, D) \equiv \omega(x, U, D)$;

3. If is $U \subset D$ an open set and $U = \bigcup_{j=1}^{\infty} K_j$, where $K_j \subset K_{j+1}^0$ are compact, then $\omega^*(x, K_j, D) \downarrow \omega^*(x, U, D)$. If is $E \subset D$ an arbitrary set, then there exists a decreasing sequence of open sets $U_j \supset E$, $U_j \supset U_{j+1}$ ($j=1, 2, \dots$) such that $(\lim_{j \rightarrow \infty} \omega(x, U_j, D))^* = \omega^*(x, E, D)$.

The following theorem relates $\omega^*(x, E, D) \equiv 0$ to m_{cv} -polar sets.

Theorem 3. $\mathcal{P}_{m_{cv}}$ -measure $\omega^*(x, E, D)$ is either nowhere equal to zero or is identically equal to zero. $\omega^*(x, E, D) \equiv 0$. Moreover, $\omega^*(x, E, D) \equiv 0$ if and only if E m_{cv} -polar in D .

Indeed, if m -convex function $\omega^*(x, E, D)$ vanishes at some interior point, $x^0 \in D$, then by the maximum principle $\omega^*(x, E, D) \equiv 0$.

Furthermore, if $\omega^*(x, E, D) \equiv 0$, then it is easy to find a point $x^0 \in D$ such that $\omega(x^0, E, D) = 0$. According to the definition,

$$0 = \omega(x^0, E, D) = \sup\{u(x^0) : u \in \mathcal{U}(E, D)\} \text{ there is a sequence } u_j \in \mathcal{U}(E, D) : u_j(x^0) > -\frac{1}{2^j}.$$

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Let's put $w(x) = \sum_{j=1}^{\infty} u_j(x)$. Since, all $u_j < 0$, that $w(x) \in m-cv(D)$. Since $u_j(x^0) > -\frac{1}{2^j}$,



that $w(x^0) > \sum_{j=1}^{\infty} \frac{1}{2^j} = -1$, i.e. $w(x) \not\equiv -\infty$. Since that $u_j|_E = -1$, From $w(x) \equiv -\infty, \forall x \in E$. This it follows that E is a mcv -polar set in D .

Conversely, if E is a mcv -polar set, D , then according to Theorem 2 there exists a function $u(x) \in m-cv(D)$, $u|_D < 0$, $u \not\equiv -\infty$, but $u|_E \equiv -\infty$. Then the functions $u_j(x) = \frac{1}{j} u(x) \in \mathcal{U}(E, D)$, $j = 1, 2, \dots$ and $\omega(y, E, D) = \sup\{u(y) : u \in \mathcal{U}(E, D)\} = 0$ for all $y \in D : u(y) \neq -\infty$. However, the set $\{y \in D : u(y) = -\infty\}$ has Lebesgue measure zero, i.e., it is nowhere dense. Therefore, $\omega^*(x, E, D) = \lim_{y \rightarrow x} \overline{\omega(y, E, D)} \equiv 0$. The theorem is proved.

4. (two constants theorem). If in the area $D \subset \mathbb{R}^n$ function $u(x) \in m-cv$ and $u|_D \leq R$, $u|_E \leq r$, ($E \subset D$, $r < R$), then for all $x \in D$ the inequality holds $u(x) \leq R(1 + \omega^*(x, E, D)) - r\omega^*(x, E, D)$.

5. If a set E compactly lies in a strongly mcv convex domain, $D = \{\rho(x) < 0\}$, $E \Subset D$, then the \mathcal{P}_{mcv} -measure $\omega^*(x, E, D)$ $m-cv$ continues in.

Definition 3. A point is called regular point of a compact set if. A compact set is called ω -regular compact set if each of its points is ω -regular.

The following theorem is very important in the study of functions.

Theorem 4. If is a regular compact set, then ω -measure is ω and is a continuous function in.

The following theorem is key in the characterization of \mathcal{P}_{mcv} -capacity.

Theorem 5. The quantity $\mathcal{P}_{mcv}(E, D)$ is an increasing and countably subadditive function of the set: $\mathcal{P}_{mcv}(E_1, D) \leq \mathcal{P}_{mcv}(E_2, D)$ for $E_1 \subset E_2$ and

$$\mathcal{P}_{mcv}\left(\bigcup_{j=1}^{\infty} E_j, D\right) \leq \sum_{j=1}^{\infty} \mathcal{P}_{mcv}(E_j, D). \quad (4)$$

Moreover, $\mathcal{P}_{mcv}(E, D)$ it is continuous on the right, i.e. for any set $E \subset D$ and any $\varepsilon > 0$ there exists an open set $U \supset E$ such that $\mathcal{P}_{mcv}(U, D) - \mathcal{P}_{mcv}(E, D) < \varepsilon$.

Proof. Monotone $\mathcal{P}_{mcv}(E, D)$ follows obviously from the monotonicity property of the \mathcal{P}_{mcv} -measure. The proof of (4) follows from a similar inequality

$$-\omega\left(x, \bigcup_{j=1}^{\infty} E_j, D\right) \leq -\sum_{j=1}^{\infty} \omega(x, E_j, D) \quad \text{for } \mathcal{P}_{mcv} \text{ -measures: for any set,}$$

$u_j(x) \in \mathcal{U}(E_j, D)$ the sum $\sum_{j=1}^{\infty} u_j(x)$ is $m-cv$ function in the wide sense (it can also



be equal to $-\infty$). In addition $\sum_{j=1}^{\infty} u_j(x) \in \mathcal{U}\left(\bigcup_{j=1}^{\infty} E_j, D\right)$, and, therefore

$\sum_{j=1}^{\infty} u_j(x) \leq \omega\left(x, \bigcup_{j=1}^{\infty} E_j, D\right)$. On the other hand,

$$\begin{aligned} & \sup\left\{\sum_{j=1}^{\infty} u_j(x) : u_j(x) \in \mathcal{U}(E_j, D)\right\} = \\ & = \sum_{j=1}^{\infty} \sup\{u_j(x) : u_j(x) \in \mathcal{U}(E_j, D)\} = \sum_{j=1}^{\infty} \omega(x, E_j, D) \end{aligned}$$

and

$$\sum_{j=1}^{\infty} \omega(x, E_j, D) \leq \omega\left(x, \bigcup_{j=1}^{\infty} E_j, D\right).$$

Integrating this inequality and using Levi's theorem on the integration of a monotone sequence, we obtain

$$-\int \omega\left(x, \bigcup_{j=1}^{\infty} E_j, D\right) dV \leq -\sum_{j=1}^{\infty} \int \omega(x, E_j, D) dV.$$

It remains to show the continuity of the set function on the right $\mathcal{P}_{mcv}(E, D)$. We fix a set $E \subset D$ and, according to the property 3) of the \mathcal{P}_{mcv} -measure, construct a

sequence of open sets $U_j \supset E$, $U_j \supset U_{j+1}$: $\left[\lim_{j \rightarrow \infty} \omega(x, U_j, D)\right]^* \equiv \omega^*(x, E, D)$.

Since, $\omega(x, U_j, D)$ is increasing, then again by Levy's theorem

$$\begin{aligned} \lim_{j \rightarrow \infty} \mathcal{P}_{mcv}(U_j, D) &= -\lim_{j \rightarrow \infty} \int \omega(x, U_j, D) dV = -\int \lim_{j \rightarrow \infty} \omega(x, U_j, D) = \\ &= -\int \left[\lim_{j \rightarrow \infty} \omega(x, U_j, D)\right]^* dV = \mathcal{P}_{mcv}(E, D). \end{aligned}$$

Hence, for any $\varepsilon > 0$, there exists j_0 such that for $j \geq j_0$, the inequality holds $\mathcal{P}_{mcv}(U_j, D) - \mathcal{P}_{mcv}(E, D) < \varepsilon$. The theorem is proved.

Consequence 1. For any decreasing sequence of compact sets, $K_1 \supset K_2 \supset \dots$ the following right-hand continuity holds:

$$\mathcal{P}_{mcv}\left(\bigcap_{j=1}^{\infty} K_j, D\right) = \lim_{j \rightarrow \infty} \mathcal{P}_{mcv}(K_j, D). \quad (5)$$

Consequence 2. For any increasing sequence of sets $E_1 \subset E_2 \subset \dots$, $E = \bigcup_{j=1}^{\infty} E_j$,

there is continuity on the left



$$\mathcal{P}_{mcv} \left(\bigcup_{j=1}^{\infty} E_j, D \right) = \lim_{j \rightarrow \infty} \mathcal{P}_{mcv} (E_j, D). \quad (6)$$

Theorem 6. *The set function $\mathcal{P}_{mcv}(E, D)$ has all the properties of Choquet measurability, and therefore any Borel set is measurable in \mathcal{P}_{mcv} -capacity. Thus, if $E \subset D$ a Borel set, $E \in B$, is measurable in capacity, then its internal and external capacities coincide: $\mathcal{P}_{*mcv}(E, D) = \mathcal{P}_{mcv}^*(E, D) = \mathcal{P}_{mcv}(E, D)$, where $\mathcal{P}_{*mcv}(E, D) = \sup \{ \mathcal{P}_{mcv}(K, D) : K \subset E - \text{compact} \}$ – internal and $\mathcal{P}_{mcv}^*(E, D) = \inf \{ \mathcal{P}_{mcv}(U, D) : U \supset E - \text{open} \}$ – external capacity of a set E .*

8. CONCLUSION

In conclusion, this paper significantly advances the mathematical framework of the theory of m -convex functions by establishing rigorous metric and capacitive characteristics for geometric domains in D . The structural analysis leads to several key findings:

Ultimately, these results offer essential tools for solving Dirichlet problems, evaluating uniform estimates of m -convex functions, and understanding the metric topology of subsets within m -convex domains.

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